The spectrum of MHD flows about X-points.

Nijboer, R.J.; Goedbloed, J.P.; Lifschitz, A.E.

published in
Journal of Plasma Physics
1998

DOI (link to publisher)
10.1017/S0022377898006941

document version
Publisher's PDF, also known as Version of record

J. Plasma Physics (1998), *vol.* 60, *part* 2, *pp.* 421–446. Printed in the United Kingdom

The spectrum of MHD flows about X points

R. J. NIJBOER,[1] J. P. GOEDBLOED[1] and A. E. LIFSCHITZ[2]

[1]FOM–Institute for Plasma Physics Rijnhuizen, Edisonbaan 14, Postbus 1207, 3430 BE Nieuwegein, The Netherlands

[2]Department of Mathematics, Statistics and Computer Science, University of Illinois, Chicago, IL 60607, USA

(Received 16 January 1998 and in revised form 11 March 1998)

A recently proposed method to calculate the spectrum of linear, incompressible, unbounded plasma flows is applied to magnetohydrodynamic flows about X points. The method transforms the two-dimensional spectral problem in physical space into a one-dimensional problem in Fourier space. The latter problem is far easier to solve. Application of this method to X-point plasma flows results in two kinds of essential spectra. One kind corresponds to stable perturbations and the other one to perturbations that become overstable whenever the square of the poloidal Alfvén Mach number becomes larger than 1. Apart from these two spectra, no other spectral values were found.

1. Introduction

In this paper we consider incompressible, plasma flows with a linear spatial dependence around an X point. As a model for the X-point geometry we consider field lines that form hyperbolas with inflow of plasma at one end and outflow at the other. In this way, we may model X points in a wide range of applications ranging from fusion to astrophysical plasmas. We are interested in the spectrum of these two-dimensional geometries with concave field lines. Recall that the concavity of the field lines in a cusp geometry is very stabilizing for the case of a static plasma, which led to high hopes in the early days of fusion research (Bernstein *et al.* 1958; Spalding 1971). On the other hand, the spectrum for the case of an X-point flow without magnetic field was calculated by Lifschitz (1997 *b*), and it was shown that these flows are always unstable. Here we investigate the spectrum of the combined problem of a magnetized plasma with flow in an X-point geometry.

The introduction of an equilibrium flow in the magnetohydrodynamic (MHD) spectral problem has major consequences. A number of properties that hold for static plasmas are lost when equilibrium flow is introduced. First of all, a static equilibrium gives rise to a self-adjoint spectral problem. This property is lost when flow is introduced. This means that the spectral values are no longer restricted to the axes of the complex plane, but may be found in the whole plane. Secondly, an energy principle, as for static plasmas, does not exist for plasmas with equilibrium flow. Although Berge (1997) recently described necessary and sufficient conditions for the stability of stationary plasmas using a Lyapunov functional, the application of these conditions to normal-mode solutions is not straightforward. His results for normal modes differ somewhat from the sufficient condition by Frieman and

Rotenberg (1960), which is explained by the fact that the concept of marginal stability loses its meaning when equilibrium flow is introduced. Hence, with the introduction of equilibrium flow, a number of properties are lost compared with the static-plasma case.

As stated, the X-point geometry considered has only one symmetry, and hence gives rise to a two-dimensional spectral problem. Most of these spectral problems have to be solved numerically. There are, however, some analytical results. One of these is on static, cylindrical plasmas with elliptical cross sections by Dewar *et al.* (1974). They used a variational approach, which works for static plasmas, but cannot be applied for plasmas with flow because of the difficulties concerning the energy principle.

Another possibility of obtaining analytical results for two-dimensional spectral problems is by considering the continuous spectrum (Hellsten and Spies 1979; Hameiri and Hammer 1979). This part of the spectrum is described largely by variations within one magnetic flux surface. This reduces the spectral problem to a one-dimensional one, which makes it tractable analytically.

In this paper we use an altogether different method for calculating two-dimensional spectral problems. It was proposed for spectral analysis by Lifschitz (1995, 1997 *a*, *b*) and based on stability analyses in fluid dynamics (Lagnado *et al.* 1984; Bayly 1986; Sahli *et al.* 1997). The basic assumptions for this method are that the flow is linear, i.e. has a linear spatial dependence, unbounded and incompressible. The method consists in solving the spectral problem in Fourier space. Since the flow is unbounded, one can apply a complete spatial Fourier transform to the spectral problem. Then, one can eliminate the perturbed total pressure and, since the flow is linear, retrieve an ordinary differential equation for the spectral problem. Hence one finds a one-dimensional spectral problem in Fourier space.

The results of this Fourier method for plasma flows with elliptical cross-sections (Lifschitz 1995, 1997 *a*) can be compared with results from the Hain–Lüst equation with flow in the limit of a circular plasma. The discrete spectrum as found from the Hain–Lüst equation lies on the same curves as the spectrum found from the Fourier method (compare results from Nijboer *et al.* (1997 *a*) with Lifschitz (1997 *a*)). However, the spectrum from the Fourier method forms a continuous curve, because the domain is unbounded. Hence, using the Fourier method, the local structure of the discrete spectrum of the finite domain (e.g. the clustering of the discrete spectrum towards the continua) is lost. On the other hand, the global structure of the spectrum (i.e. the shape of the spectral curves) is retrieved. Furthermore, whenever the plasma flow is really two-dimensional, the Fourier method reduces to solving a one-dimensional problem, whereas a method in physical space would require the solution of a two-dimensional problem. Hence the Fourier method leads to a much simpler spectral problem.

Application of the Fourier method to X-point flows without magnetic field leads to the surprising result that the spectrum consists of a strip of spectral values (Lifschitz 1997 *b*). In order to investigate the spectrum of these X-point geometries further, we consider in this paper the extended problem of a magnetized X-point plasma flow. In Sec. 2 we consider the equilibrium. The modified Grad–Shafranov equation for incompressible-flow equilibria is derived, and with this formalism X-type equilibria are described. Then in Sec. 3 the equilibrium is perturbed and the spectral problem is formulated. In Sec. 4 the spectrum of azimuthal modes is considered, and in Sec. 5 that of general modes. This spectrum is calculated in Fourier space. The eigenfunctions in physical space are related to those in Fourier space by the Fourier transform. In Sec. 6 this relation is made explicit for modes belonging to the 'classical' continuous spectrum. We end this paper with conclusions in Sec. 7.

2. Equilibrium state

The equilibrium state of incompressible MHD flows is described by the modified Grad–Shafranov equation. In this section this equation is derived from the MHD equations and used to describe linear plasma flows about X points. The MHD equations describing the behaviour of a plasma are

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{V}) = 0, \tag{2.1a}$$

$$\rho(\partial_t \mathbf{V} + \mathbf{V} \cdot \boldsymbol{\nabla}\mathbf{V}) + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \boldsymbol{\nabla} P = 0, \tag{2.1b}$$

$$\partial_t \mathbf{B} - \boldsymbol{\nabla} \times (\mathbf{V} \times \mathbf{B}) = 0, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{2.1c}$$

where ρ is the plasma density, $\mathbf{V}$ is the plasma flow, $\mathbf{B}$ is the magnetic field and P is the kinetic pressure. In order to close the system of equations, it is assumed that the plasma is incompressible:

$$\boldsymbol{\nabla} \cdot \mathbf{V} = 0. \tag{2.2}$$

Consider two-dimensional, stationary equilibria, which are translationally symmetric along the z axis (Agim and Tataronis 1985). This means that in equations (2.1) $\partial_t = 0$ and $\partial_z = 0$. It then follows from the continuity equation and the divergence-free condition of the magnetic field that the velocity field and the magnetic field can be written as

$$\mathbf{V} = \rho^{-1}\mathbf{e}_z \times \boldsymbol{\nabla}\chi + V_z\mathbf{e}_z, \tag{2.3a}$$

$$\mathbf{B} = \mathbf{e}_z \times \boldsymbol{\nabla}\psi + B_z\mathbf{e}_z. \tag{2.3b}$$

From the poloidal component of Faraday's law it is then found that $\chi = \chi(\psi)$. Furthermore, the divergence-free condition of the velocity field yields $\rho = \rho(\psi)$ and the longitudinal components of the momentum equation and Faraday's law yield $V_z = V_z(\psi)$ and $B_z = B_z(\psi)$.

This leaves the poloidal component of the momentum equation. From the component perpendicular to $\boldsymbol{\nabla}\psi$, one finds $H = H(\psi)$, where

$$H(\psi) = P + \tfrac{1}{2}\rho V_p^2 + \tfrac{1}{2}B_z^2 \tag{2.4}$$

is a Bernoulli type of function and V_p is the poloidal component of the velocity field. Finally, the component parallel to $\boldsymbol{\nabla}\psi$ yields the modified Grad–Shafranov equation

$$(1 - M^2)\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\psi - \tfrac{1}{2}(M^2)'|\boldsymbol{\nabla}\psi|^2 + H' = 0, \tag{2.5}$$

where $' \equiv d/d\psi$ and $M^2 \equiv \rho V_p^2/B_p^2 = \chi'^2/\rho$ is the square of the poloidal Alfvén Mach number. Equation (2.5) determines the flux ψ when M^2 and H are prescribed as functions of ψ. Only the last two of the five free functions ρ, V_z, B_z, M and H determine the flux ψ. This is due to the translational symmetry. In general the flux depends on more of these functions (Agim and Tataronis 1985).

Note that, since ρ is a flux function, M^2 is also a flux function. For compressible flows, this is not the case and the perpendicular component of the momentum equation yields an equation relating M to ψ and $\nabla\psi$ (Lifschitz and Goedbloed 1997; Goedbloed and Lifschitz 1997). This is the Bernoulli equation, which then, together with the modified Grad–Shafranov equation, determines the flux and the poloidal Alfvén Mach number. In this compressible case the modified Grad–Shafranov equation is a partial differential equation of mixed elliptc/hyperbolic type. Here the flow is incompressible, and therefore the modified Grad–Shafranov equation is always of elliptic type.

For the equilibrium profiles, the following choice is made. M is taken to be a constant and the Bernoulli function is taken to be linear in ψ,

$$H = C - (1 - M^2) j_z \psi, \tag{2.6}$$

where the longitudinal current density j_z is constant. This yields the following equation for the magnetic flux:

$$\nabla \cdot \nabla\psi = j_z. \tag{2.7}$$

Equation (2.7) describes both flux surfaces that have ellipses as projections and flux surfaces that have hyperbolas as projections. The elliptical equilibria were described by Gajewski (1972) and their spectral stability was analysed by Lifschitz (1995, 1997*a*). Here the case of flux surfaces that have hyperbolas as projections is considered (Fig. 1).

Analogous to Gajewski (1972), we could consider an 'inner' region filled with plasma and an 'outer' vacuum region. Since this illustrates a relevant extension of the model, we shall briefly describe the equilibrium features, although the vacuum will be ignored in the spectral analysis. Inside the inner region there is a constant current density j_z, and in the outer region the current density is zero. When the plasma vacuum interface is a hyperbola of constant flux ψ_0, the inner solution is

$$\psi = \psi_0 \left[\left(\frac{x}{a_1} \right)^2 - \left(\frac{y}{a_2} \right)^2 \right], \tag{2.8}$$

where a_1 and a_2 are constants determining the shape of the hyperbola, and

$$j_z = 2\psi_0 \left(\frac{1}{a_1^2} - \frac{1}{a_2^2} \right). \tag{2.9}$$

When $a_1 = a_2$, the surfaces form 'perfect' hyperbolas and the current density j_z is zero. The flux surfaces for this situation are drawn in Fig. 1. Whenever $a_1 \neq a_2$, the two separatrices are not perpendicular to each other, and the flux surfaces are modified appropriately.

The shape of the geometry can be expressed in a dimensionless parameter

$$\delta = \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2}, \tag{2.10}$$

which is related to the angle ϑ between the two separatrices: $\delta = \tan(\frac{1}{2}\vartheta - \frac{1}{4}\pi)$. The value of δ ranges from -1 to 1, where the two separatrices coincide for $\delta = 1$ and $\delta = -1$ and are perpendicular for $\delta = 0$. In the latter case, we call the hyperbola 'perfect'.

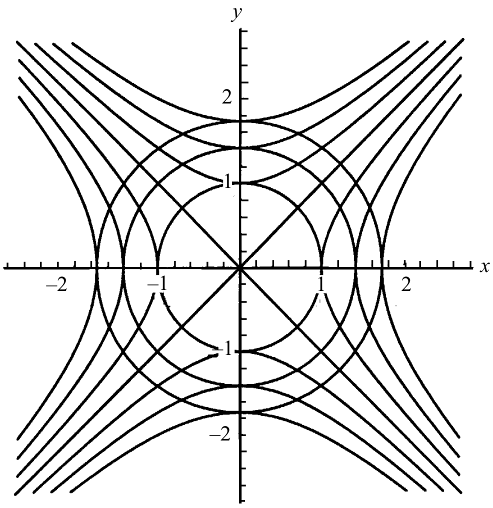

Figure 1. The separatrices, projected flux surfaces (hyperbolas) and isobars of the total pressure (circles) for a 'perfect' hyperbolic geometry ($\delta = 0$).

Demanding continuity of the total pressure at the plasma–vacuum interface yields the boundary condition

$$\left.\frac{\partial \psi}{\partial n}\right|_{\text{ext}} = (1 - M^2)^{1/2} \left.\frac{\partial \psi}{\partial n}\right|_{\text{int}} \tag{2.11}$$

for the special case that $H(\psi_0) = \frac{1}{2} B_{z,\text{ext}}^2$ and $M^2 < 1$. In this case, the flux function in the vacuum region of the quadrant $x > 0$, $y^2 < a_2^2 x^2 / a_1^2$ (other quadrants can be treated similarly) is

$$\psi = \psi_0 \left\{ 1 - \left(\frac{1 - M^2}{1 - \delta^2} \right)^{1/2} [2\delta(u - u_0) + \sin 2(u - u_0) \cosh 2v] \right\}, \tag{2.12}$$

where orthogonal coordinates u and v have been introduced:

$$x = (a_1^2 + a_2^2)^{1/2} \cos u \cosh v, \qquad y = (a_1^2 + a_2^2)^{1/2} \sin u \sinh v. \tag{2.13}$$

In these coordinates, $v = \text{const}$ describes ellipses in the (x, y) plane and $u = \text{const}$ describes hyperbolas. The plasma–vacuum interface is described by the hyperbola $u = u_0$, where u_0 is determined by the geometry: $u_0 = \frac{1}{2} \arccos \delta$. The vacuum region is described by $0 \leqslant u \leqslant u_0$ and $-\infty < v < \infty$.

Although the solution (2.12) is regular in the (u, v) variables, it gives rise to a singularity in the (x, y) variables. For $u = 0$, we have $y = 0$ and $x = (a_1^2 + a_2^2)^{1/2} \cosh v \geqslant (a_1^2 + a_2^2)^{1/2}$. On this line, we have

$$\left.\frac{\partial \psi}{\partial y}\right|_{u=0} = -4 \frac{\psi_0}{(a_1^2 + a_2^2)^{1/2}} \frac{\delta}{(1 - \delta^2)^{1/2}} \sinh v. \tag{2.14}$$

Hence, whenever $\delta \neq 0$, $\partial \psi / \partial y$ is discontinuous at $u = 0$. This means that B_x is discontinuous and hence that a current sheet exists at $u = 0$. The reason for this singularity is that, at the plasma–vacuum interface, ψ and its normal derivative are prescribed. This is one boundary condition too much, which gives rise to the described singularity.

In the rest of this paper, no vacuum region is taken into account so that the plasma extends up to infinity in all directions. The magnetic flux is then given by (2.8) alone. When the profiles ρ, V_z and B_z are taken to be constant, the equilibrium is completely specified:

$$\rho(x, y) = \rho, \tag{2.15a}$$

$$\mathbf{V}(x, y) = \frac{\alpha}{\rho^{1/2}} \left(\frac{a_1}{a_2} y\mathbf{e}_x + \frac{a_2}{a_1} x\mathbf{e}_y \right) + V_z \mathbf{e}_z, \tag{2.15b}$$

$$\mathbf{B}(x, y) = \beta \left(\frac{a_1}{a_2} y\mathbf{e}_x + \frac{a_2}{a_1} x\mathbf{e}_y \right) + B_z \mathbf{e}_z, \tag{2.15c}$$

$$P(x, y) = C - \tfrac{1}{2} B_z^2 - \tfrac{1}{2} \left[\left(\alpha^2 - \frac{2\delta}{1+\delta} \beta^2 \right) x^2 + \left(\alpha^2 - \frac{2\delta}{1-\delta} \beta^2 \right) y^2 \right]. \tag{2.15d}$$

The constants α and β are defined as

$$\alpha = \frac{2}{a_1 a_2} M \psi_0, \qquad \beta = \frac{2}{a_1 a_2} \psi_0. \tag{2.16}$$

Note that $M = \alpha/\beta$. This notation allows us in the remainder of the paper to take the limits of hydrodynamics ($\beta = 0$) and static MHD ($\alpha = 0$) in a straightforward manner. In the hydrodynamic limit, $\psi_0 \to 0$, whereas at the same time $M \to \infty$ such that α remains bounded. In the static MHD limit, $M \to 0$, whereas ψ_0 remains finite.

From the equilibrium profiles (2.15) the following expressions for the vorticity $\boldsymbol{\omega}$, the electric current $\mathbf{j}$ and the total pressure Π can be derived:

$$\boldsymbol{\omega}(x, y) = \nabla \times \mathbf{V} = -\frac{\alpha}{\rho^{1/2}} \left(\frac{a_1}{a_2} - \frac{a_2}{a_1} \right) \mathbf{e}_z, \tag{2.17a}$$

$$\mathbf{j}(x, y) = \nabla \times \mathbf{B} = -\beta \left(\frac{a_1}{a_2} - \frac{a_2}{a_1} \right) \mathbf{e}_z, \tag{2.17b}$$

$$\Pi(x, y) = C - \tfrac{1}{2}(\alpha^2 - \beta^2)(x^2 + y^2). \tag{2.17c}$$

These expressions show that whenever $a_1 = a_2$ the vorticity and the current density are identically zero. Furthermore, isobars of the total pressure always form circles, independent of the geometry. The constant C equals the total pressure 'on axis', i.e. $C = \Pi(0, 0)$.

It is noted that a number of equilibrium quantities become infinite for x or y going to infinity. The total pressure Π and the gas pressure P may even become negative. However, for every bounded domain, these equilibrium quantities are finite, and we can choose an appropriate value for C such that both pressures are positive in this domain. In the rest of this paper we disregard the problem, demanding that the perturbations, which are introduced in the next section, fall off sufficiently fast for x and y going to infinity. This guarantees that the energy of these perturbations remains finite.

3. Incompressible perturbations

In this section the spectral problem is formulated, similar to the procedure of Lifschitz (1995, 1997 *a*, *b*). The equilibrium (2.15) is perturbed, exploiting the total pressure Π instead of the kinetic pressure P:

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{v}, \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \quad \Pi = \Pi_0 + \pi, \quad \rho = \rho_0 + \rho_1. \tag{3.1}$$

Then (2.1) are linearized and made dimensionless by applying the following scaling:

$$x' = \frac{x}{a_1}, \quad y' = \frac{y}{a_2}, \quad z' = \frac{z}{a_3}, \quad t' = \frac{t}{T}, \tag{3.2a}$$

$$\rho' = \frac{\rho_1}{\rho_0}, \quad \pi' = \frac{T^2\pi}{\rho_0 a_3^2}, \quad v_i' = \frac{Tv_i}{a_i}, \quad b_i' = \frac{Tb_i}{\rho_0^{1/2} a_i} \quad (i = 1, 2, 3), \tag{3.2b}$$

where

$$a_3 \equiv [\tfrac{1}{2}(a_1^2 + a_2^2)]^{1/2}, \qquad T \equiv \frac{a_1 a_2 \rho_0^{1/2}}{2(1+M^2)^{1/2}|\psi_0|}. \tag{3.3}$$

The equilibrium quantities are made dimensionless in a similar fashion:

$$\alpha' = \frac{T\alpha}{\rho_0^{1/2}}, \qquad \beta' = \frac{T\beta}{\rho_0^{1/2}}, \qquad B_z' = \frac{TB_z}{\rho_0^{1/2} a_3}, \qquad V_z' = \frac{TV_z}{a_3}. \tag{3.4}$$

Note that the x and y directions as well as the x and y vector components are scaled differently. In this way, the geometry is transformed to that of a 'perfect' hyperbola, which simplifies the derivations. However, one should be careful when considering lengths or inner products, since this scaling has also changed the metric.

The primes are dropped below for the sake of brevity. Assuming an exponential time dependence, $f(\mathbf{x}, t) = e^{\sigma t} f(\mathbf{x})$, the above linearization and scaling yields the following spectral problem:

$$\sigma\rho + (\alpha\mathscr{D} + V_z\partial_z)\,\rho = 0, \tag{3.5a}$$

$$\sigma\mathbf{v} + (\alpha\mathscr{D} + V_z\partial_z)\,\mathbf{v} - (\beta\mathscr{D} + B_z\partial_z)\,\mathbf{b} + \alpha\mathscr{T}\mathbf{v} - \beta\mathscr{T}\mathbf{b} + \alpha^2\mathbf{x}_\perp\rho + \mathscr{E}\pi = 0, \tag{3.5b}$$

$$\sigma\mathbf{b} + (\alpha\mathscr{D} + V_z\partial_z)\,\mathbf{b} - (\beta\mathscr{D} + B_z\partial_z)\,\mathbf{v} + \beta\mathscr{T}\mathbf{v} - \alpha\mathscr{T}\mathbf{b} = 0, \tag{3.5c}$$

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0, \qquad \boldsymbol{\nabla} \cdot \mathbf{b} = 0. \tag{3.5d}$$

Here the subscript $\perp$ denotes projection onto the (x, y) plane and

$$\mathscr{D} = \mathscr{T}\mathbf{x} \cdot \boldsymbol{\nabla} = y\partial_x + x\partial_y, \qquad \mathscr{T} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathscr{E} = \frac{\mathbf{e}_x}{1+\delta}\partial_x + \frac{\mathbf{e}_y}{1-\delta}\partial_y + \mathbf{e}_z\partial_z.$$

The operator $\mathscr{D}$ denotes differentiation along the projection of the flux surfaces. It originates from the $\mathbf{B}_0 \cdot \boldsymbol{\nabla}$ and $\mathbf{V}_0 \cdot \boldsymbol{\nabla}$ operators, but it is independent of the geometry because of the different scaling of the x and the y direction; see (3.2). Owing to this scaling, all geometrical effects are transformed into the $\mathscr{E}$ operator.

At this point, we note that (3.5) has the same form as for elliptical flux surfaces (Lifschitz 1997 *a*). The only difference is the operator $\mathscr{T}$, which in the elliptic case is rotation by 90° in the (x, y) plane. In the rest of our derivations this will give rise to some differences in plus and minus signs. However, although the equations may seem similar, the problem for hyperbolic flux surfaces is quite different from that for elliptical flux surfaces. This is due to the fact that for the hyperbolic case the flux surfaces are open, giving rise to boundary conditions at infinity, whereas in the elliptic case the flux surfaces are closed, yielding periodic boundary conditions. This means that, apart from having to apply different techniques to solve the problem, also quite different results may be expected.

Starting to rework (3.5) we note that whenever $\sigma + \alpha\mathscr{D} + V_z\partial_z \neq 0$ it follows that $\rho = 0$, and hence that there is no perturbation of the density. In the rest of this paper we assume that this is the case. Introducing the incompressible Lagrangian displacement vector $\boldsymbol{\xi}$,

$$\mathbf{v} \equiv \partial_t\boldsymbol{\xi} + \mathbf{V}_0 \cdot \boldsymbol{\nabla}\boldsymbol{\xi} - \boldsymbol{\xi} \cdot \boldsymbol{\nabla}\mathbf{V}_0 = (\sigma + \alpha\mathscr{D} + V_z\partial_z - \alpha\mathscr{T})\boldsymbol{\xi}, \tag{3.6}$$

enables us to integrate the linearized Faraday's law whenever σ does not belong to the flow continuum (Spies 1978). This yields

$$\mathbf{b} = (\beta\mathscr{D} + B_z\partial_z - \beta\mathscr{T})\boldsymbol{\xi}. \tag{3.7}$$

The case where σ belongs to the flow continuum coincides with the condition that $\sigma + \alpha\mathscr{D} + V_z\partial_z = 0$ and is not considered here.

With these assumptions, the spectral problem (3.5) reduces to

$$\mathscr{H}\xi_x - (\alpha^2 - \beta^2)\xi_x + \frac{1}{1+\delta}\partial_x\pi = 0, \tag{3.8}$$

$$\mathscr{H}\xi_y - (\alpha^2 - \beta^2)\xi_y + \frac{1}{1-\delta}\partial_y\pi = 0, \tag{3.9}$$

$$\mathscr{H}\xi_z + \partial_z\pi = 0, \tag{3.10}$$

$$\partial_x\xi_x + \partial_y\xi_y + \partial_z\xi_z = 0, \tag{3.11}$$

where $\mathscr{H} = (\sigma + \alpha\mathscr{D} + V_z\partial_z)^2 - (\beta\mathscr{D} + B_z\partial_z)^2$. This is a system of four partial differential equations for four unknowns $\boldsymbol{\xi}$, π. However, when applying a spatial Fourier transform in Cartesian coordinates, we can reduce this system to a system of ordinary differential equations in Fourier space, which allows us to solve the problem (Lagnado *et al.* 1984; Lifschitz 1995, 1997 *a*, *b*). Here we make use of the fact that the domain of the problem is unbounded, which allows for a Fourier transform instead of a Fourier series.

Applying a Fourier transform in all three spatial variables,

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \hat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}, \tag{3.12}$$

transforms the spectral problem for σ into

$$\hat{\mathscr{H}}\hat{\xi}_x - (\alpha^2 - \beta^2)\hat{\xi}_x + \frac{k_x}{1+\delta}i\hat{\pi} = 0, \tag{3.13}$$

$$\hat{\mathscr{H}}\hat{\xi}_y - (\alpha^2 - \beta^2)\hat{\xi}_y + \frac{k_y}{1-\delta}i\hat{\pi} = 0, \tag{3.14}$$

$$\hat{\mathscr{H}}\hat{\xi}_z + k_z i\hat{\pi} = 0, \tag{3.15}$$

$$k_x\hat{\xi}_x + k_y\hat{\xi}_y + k_z\hat{\xi}_z = 0, \tag{3.16}$$

where

$$\hat{\mathscr{H}} = (\sigma - \alpha\hat{\mathscr{D}} + ik_zV_z)^2 - (\beta\hat{\mathscr{D}} - ik_zB_z)^2, \qquad \hat{\mathscr{D}} = k_x\partial_{k_y} + k_y\partial_{k_x}.$$

The Fourier transform in Cartesian coordinates transforms all ∂_is into scalars, whereas $\mathscr{D}$ is transformed into $-\hat{\mathscr{D}}$. In this way the problem can be reduced to a system of ordinary differential equations, whereas a transformation in other coordinates (e.g. flux coordinates) would not reduce to ordinary differential equations, but would retain a system of partial differential equations.

Performing a Fourier transform in Cartesian coordinates, however, does not restrict us from applying coordinate transformations in physical and Fourier space. Hence, when introducing hyperbolic coordinates in these spaces,

$$x = r\cosh\phi, \qquad y = r\sinh\phi, \tag{3.17a}$$

$$k_x = \rho\cosh\varphi, \qquad k_y = \rho\sinh\varphi, \tag{3.17b}$$

simple expressions for the operators $\mathscr{D}$ and $\hat{\mathscr{D}}$ are found:

$$\mathscr{D} = \partial_\phi, \qquad \hat{\mathscr{D}} = \partial_\varphi. \tag{3.18}$$

In this way, the spectral problem in Fourier space is reduced to a system of ordinary differential equations, which are far easier to solve than the original system of partial differential equations. The Fourier transform in these hyperbolic coordinates then reads

$$f(r,\phi,z) = \frac{1}{(2\pi)^{3/2}}\int \hat{f}(\rho,\varphi,k_z)e^{ir\rho\cosh(\phi+\varphi)+ik_zz}\rho\, d\rho\, d\varphi\, dk_z. \tag{3.19}$$

Note that the described coordinate transformation is valid in the quadrants $x > |y|$ and $k_x > |k_y|$. However, similar transformations apply for the other quadrants.

In order to complete the spectral problem, regularity conditions have to be specified. The appropriate conditions for the perturbations are that they are square-integrable. This condition is the same in physical space and in Fourier space, and implies that all perturbations will have finite energy. It means that the eigenfunctions will fall off at infinity. It is good to note that discrete modes (if they exist) satisfy this condition, but modes belonging to the essential spectrum may violate it. However, the latter modes can be found as the limit of functions that do satisfy the regularity conditions. This means that modes belonging to the essential spectrum may remain non-vanishing but bounded at infinity, and therefore may have infinite energy. For more details, the reader is referred to Lifschitz (1989).

It follows from (3.13)–(3.16) that V_z only gives rise to a constant Doppler shift of the complete spectrum. Therefore it can be chosen to be zero without loss of generality. Also, hats will be dropped for the sake of brevity.

3.1. Azimuthal modes

For azimuthal modes ($k_z = 0$), the spectral problem (3.13)–(3.16) simplifies considerably. For these modes, ξ_z decouples from the other variables and is determined from (3.15), which in hyperbolic coordinates transforms to

$$\left[(\alpha^2-\beta^2)\frac{d^2}{d\varphi^2} - 2\alpha\sigma\frac{d}{d\varphi} + \sigma^2\right]\xi_z = 0. \tag{3.20}$$

The perturbed total pressure can be expressed in terms of the displacement vector by taking combinations of (3.13) and (3.14):

$$\frac{c^2+s^2-\delta}{1-\delta^2}\rho i\pi = -c\mathscr{H}\xi_x - s\mathscr{H}\xi_y + (\alpha^2-\beta^2)(c\xi_x+s\xi_y)$$
$$= 2(\alpha^2-\beta^2)(s\mathscr{D}\xi_x + c\mathscr{D}\xi_y) - 2(\alpha\sigma - i\beta k_z B_z)(s\xi_x + c\xi_y). \quad (3.21)$$

Here we have introduced the abbreviations

$$c \equiv \cosh\varphi, \qquad s \equiv \sinh\varphi. \quad (3.22)$$

and made use of the incompressibility condition (3.16).

A different combination of (3.13) and (3.14) eliminates π and forms together with (3.16) a system of two equations for ξ_x and ξ_y. Equation (3.16) is an algebraic relation between ξ_x and ξ_y, and is satisfied by introducing a new variable γ:

$$\xi_x = \frac{-s}{(c^2+s^2-\delta)^{1/2}}\gamma, \qquad \xi_y = \frac{c}{(c^2+s^2-\delta)^{1/2}}\gamma. \quad (3.23)$$

This results in the following eigenvalue equation for γ:

$$\left[(\alpha^2-\beta^2)\frac{d^2}{d\varphi^2} - 2\alpha\sigma\frac{d}{d\varphi} + \sigma^2 - (\alpha^2-\beta^2)f_\delta(\varphi)\right]\gamma = 0, \quad (3.24)$$

where the function f_δ is defined by

$$f_\delta(\varphi) = 1 + \frac{1-\delta^2}{(c^2+s^2-\delta)^2}. \quad (3.25)$$

In summary, the spectral problem for azimuthal modes is given by (3.20) and (3.24) for ξ_z and γ respectively. The problem is completed by considering appropriate regularity conditions, i.e. ξ_z and γ should be square-integrable. The solutions for ξ_x, ξ_y and π follow from those for ξ_z and γ. This spectral problem for σ depends on the poloidal Mach number $M = \alpha/\beta$ and the parameter δ, which describes the geometry. It does not depend on B_z and ρ. From the latter observation, it follows that the spectrum will be infinitely degenerate, since it will be the same for every value of ρ.

3.2. *Non-azimuthal modes*

For non-azimuthal modes ($k_z \neq 0$), a different approach is used to simplify the spectral problem. In this case, (3.15) and (3.16) may be used to express ξ_z and π in terms of ξ_x and ξ_y. Using (3.13) and (3.14), the expression for π can be transformed into

$$\frac{c^2+s^2-\delta+(1-\delta^2)k_z^2/\rho^2}{1-\delta^2}\rho i\pi = 2(\alpha^2-\beta^2)(c\xi_x + s\xi_y + s\mathscr{D}\xi_x + c\mathscr{D}\xi_y)$$
$$-2(\alpha\sigma - i\beta k_z B_z)(s\xi_x + c\xi_y). \quad (3.26)$$

With the expressions (3.16) for ξ_z and (3.26) for π, the spectral problem for non-azimuthal modes is described by (3.13) and (3.14). The problem is completed by demanding that all perturbations be square-integrable. This means that, apart from ξ_x and ξ_y, $c\xi_x + s\xi_y$ must also be square-integrable, since ξ_z must be square-integrable. In order to meet this last condition, a transformation of variables is performed. Define

$$\xi = \frac{c\xi_x - s\xi_y}{(c^2+s^2)^{1/2}}, \qquad \tilde{\eta} = (c^2+s^2)^{1/2}\eta = s\xi_x + c\xi_y. \tag{3.27}$$

In these variables, (3.13) and (3.14) transform into

$$\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \begin{pmatrix} \xi \\ \tilde{\eta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.28}$$

where

$$\begin{aligned}
n_{11} &= (\alpha^2-\beta^2)\frac{d^2}{d\varphi^2} - 2(\alpha\sigma - i\beta F_z)\frac{d}{d\varphi} + \sigma^2 + F_z^2 \\
&\quad -(\alpha^2-\beta^2)\left[1 + \frac{1}{(c^2+s^2)^2}\right], \\
n_{12} &= \frac{1}{(c^2+s^2)^{1/2}}\left[2(\alpha^2-\beta^2)\left(\lambda + \frac{1}{c^2+s^2}\right)\frac{d}{d\varphi}\right. \\
&\quad \left. -2(\alpha\sigma - i\beta F_z)\left(\lambda + \frac{1}{c^2+s^2}\right) - (\alpha^2-\beta^2)\frac{8cs}{(c^2+s^2)^2}\right], \\
n_{21} &= \frac{1}{(c^2+s^2)^{1/2}}\left[-2(\alpha^2-\beta^2)\frac{d}{d\varphi} + 2(\alpha\sigma - i\beta F_z) + (\alpha^2-\beta^2)\frac{4cs}{c^2+s^2}\right], \\
n_{22} &= (\alpha^2-\beta^2)\frac{d^2}{d\varphi^2} - 2\left[\alpha\sigma - i\beta F_z - (\alpha^2-\beta^2)\left(\mu - \frac{2cs}{c^2+s^2}\right)\right]\frac{d}{d\varphi} \\
&\quad +\sigma^2 + F_z^2 - 2(\alpha\sigma - i\beta F_z)\left(\mu - \frac{2cs}{c^2+s^2}\right) - (\alpha^2-\beta^2)\frac{4}{(c^2+s^2)^2},
\end{aligned}$$

and the following shorthand notation is introduced:

$$\lambda(\varphi;\delta,\kappa) \equiv \frac{1-\delta(c^2+s^2)}{c^2+s^2-\delta+(1-\delta^2)\kappa^2},$$

$$\mu(\varphi;\delta,\kappa) \equiv \frac{2cs}{c^2+s^2-\delta+(1-\delta^2)\kappa^2},$$

$$F_z \equiv k_z B_z, \qquad \kappa \equiv k_z/\rho.$$

Hence the spectral problem for non-azimuthal modes is given by (3.28). It is a system of two nonlinear ordinary differential equations that depend parametrically on $M = \alpha/\beta$, F_z, κ and δ. The problem is completed by demanding that ξ and $\tilde{\eta}$ be square-integrable.

3.3. Spectrum of two-dimensional unbounded domains

We now have the spectral problem in its final form. In the next section the spectrum for the case of azimuthal modes will be investigated and in the section thereafter for the case of non-azimuthal modes. But first we discuss here what kind of spectra one may expect to find.

When a plasma is considered in a finite domain, there will, in general, be two kinds of spectra: a continuous spectrum of singular modes connected with separate flux surfaces, and a discrete spectrum of global modes. The continuous spectrum can be retrieved by taking the limit $\rho \to \infty$ (see Goedbloed 1975). It forms a continuous curve by varying over the flux surfaces, unless every flux surface generates the same spectral point, in which case the continuous spectrum is degenerate. In the case of hyperbolic open flux surfaces, these surfaces are infinite, and hence the spectrum may become a dense set because of the additional possibility of a continuous variation of the poloidal mode number. In this way, *the spectrum becomes 'doubly' continuous*, and may form points, curves or surfaces in the complex σ plane, depending on its degeneracy.

The discrete set of spectral points that arises for a finite domain, may also become a dense set when the domain becomes infinite (see Lifschitz 1997 *a*). Since the flux surfaces are open, there are two infinite directions, and again a 'doubly' continuous spectrum arises. This may lead to a spectral set consisting of points, curves or surfaces in the complex σ plane. This again will depend on the degeneracy of the spectrum.

Apart from these continuous modes that have a clear counterpart in finite domains, in an infinite domain there may also be 'discrete' modes. These are modes for which the eigenfunctions fall off at infinity, in contrast to the modes discussed before. If they exist, they give rise to a point spectrum, which may be dense to form curves or fill surfaces. Their analogue in finite domains is not so clear, although they are reminiscent ballooning modes, which are localized on field lines of infinite length.

4. Spectrum of azimuthal modes

In this section the spectral problem for $k_z = 0$ is considered. In this case we have to solve (3.20) and (3.24) subject to the conditions that ξ_z and γ be square-integrable. We start with (3.20) for parallel displacements, ξ_z, and after that consider the more complicated equation (3.24) for perpendicular displacements γ.

4.1. Parallel displacements

Equation (3.20) for ξ_z is an ordinary differential equation with constant coefficients. Therefore it has solutions of the form

$$\xi_z = \exp(in\varphi), \quad n \in \mathbb{R}. \tag{4.1}$$

This leads to the following spectral points:

$$\sigma = in(\alpha \pm \beta). \tag{4.2}$$

Hence the plasma is stable against these perturbations.

We note that although the displacement vector remains bounded, its norm is unbounded, since the domain is unbounded. This indicates that these modes belong to the essential spectrum. Indeed the spectrum forms a continuous line along the imaginary axis, since n may vary continuously. If the domain (in Fourier space) were bounded, this would give rise to discrete values of n, and hence to a discretized spectrum. Note, however, that this would give rise to a *different* discretized spectrum than when the domain in *physical* space is bounded.

As stated in the previous section, the continuous spectrum of singular modes on flux surfaces is retrieved in the limit $\rho \to \infty$. However, for these parallel displacements the spectrum (and even the eigenfunctions) does not depend on the value of ρ. This means that the spectrum is infinitely degenerate and that it corresponds to the continuous spectrum as found for finite domains.

4.2. Perpendicular displacements

Equation (3.24) for γ is a second-order ordinary differential equation with non-constant coefficients. Here we investigate this equation and the spectrum it gives rise to. First, the special cases of $M = \infty$ (pure flow), $M = 1$ (the poloidal flow strength equals the poloidal magnetic field strength) and $M = 0$ (static plasma) are considered. The first two of these cases can be solved analytically, whereas in the third case the problem reduces to the quantum mechanical problem of a free particle moving around a potential barrier. The subsection ends by addressing the problem for general values of M.

4.2.1. Case I: pure flow.. In the case of a pure flow without magnetic field ($\alpha = 1$, $\beta = 0$; $M = \infty$), (3.24) is rewritten in terms of η; see (3.27) (note: not $\tilde{\eta}$). This results in

$$\left(\sigma - \frac{d}{d\varphi} - \frac{2cs}{c^2 + s^2 - \delta}\right)(c^2+s^2)^{1/2}(c^2+s^2-\delta)^{1/2}\left(\sigma - \frac{d}{d\varphi} - \frac{2cs}{c^2 + s^2}\right)\eta = 0. \quad (4.3)$$

The regularity condition in terms of η be the demand that $e^{2|\varphi|}\eta$ is square-integrable. Equation (4.3) can be integrated twice, and has the solution

$$\eta(\varphi) = \frac{e^{\sigma\varphi}}{(c^2 + s^2)^{1/2}}\left\{C_1 + C_2 \arctan\left[\frac{e^{2\varphi} - \delta}{(1 - \delta^2)^{1/2}}\right]\right\}. \quad (4.4)$$

By choosing appropriate values for C_1 and C_2 we can construct solutions that satisfy the boundary conditions at infinity. It can be shown that only for

$$\mathrm{Re}(\sigma) = \pm 1 \quad (4.5)$$

is there an eigenfunction for which $e^{2|\varphi|}\eta$ is bounded both at $\varphi = +\infty$ and $\varphi = -\infty$. Hence the spectrum is given by the condition (4.5).

Again these modes belong to the essential spectrum. Apart from the spectrum (4.5), there are no other modes. However, these spectral values form just the boundary of the set of spectral values as found in Lifschitz (1997 *b*), namely $|\mathrm{Re}(\sigma)| \leqslant 1$. The reason for this is that in Lifschitz (1997 *b*) it was demanded that $\mathbf{v}$ remain bounded, whereas we demand that $\boldsymbol{\xi}$ remain bounded. For pure flows, the latter condition is more restrictive than the former, resulting in a smaller set of spectral values. The eigenfunctions, on the other hand, are in full agreement.

4.2.2. Case II: flow equals magnetic field. When the poloidal flow equals the poloidal magnetic field ($\alpha = \beta$; $M = 1$) (3.24) reduces to

$$\sigma\left(\sigma - 2\alpha\frac{d}{d\varphi}\right)\gamma = 0. \quad (4.6)$$

This equation has two obvious solutions

$$\sigma = 0, \quad (4.7)$$

and

$$\gamma(\varphi) = \exp(\sigma\varphi/2\alpha). \quad (4.8)$$

Hence the boundedness of γ results in the (essential) spectrum

$$\mathrm{Re}(\sigma) = 0, \quad (4.9)$$

and in this case the plasma is stable.

4.2.3. Case III: static plasma. In the case of a static plasma ($\alpha = 0$, $\beta = 1$; $M = 0$), (3.24) takes the form

$$\frac{d^2\gamma}{d\varphi^2} - \frac{1-\delta^2}{(c^2+s^2-\delta)^2}\gamma = (\sigma^2+1)\gamma. \tag{4.10}$$

When we define $E = -(\sigma^2+1)$, (4.10) is the same as the equation for the quantum mechanical problem describing a free particle with energy E that moves around a potential barrier (Landau and Lifschitz 1977). For this problem, it is well known that the spectrum E is real, positive and continuous. Since here the boundary conditions are the same, one finds the same spectrum, resulting in

$$\sigma^2 < -1, \qquad \mathrm{Re}(\sigma) = 0. \tag{4.11}$$

Furthermore, when $\delta = 0$, it is possible to express the eigensolutions γ in terms of hypergeometric functions (Landau and Lifschitz 1977, Sec. 25):

$$\begin{aligned}\gamma(\varphi) = (c^2+s^2)^k \big[&C_1 F\left(\tfrac{1}{2}-k, \tfrac{1}{2}-k, 1-k; \tfrac{1}{2}(1-\zeta)\right) \\ &+C_2(1-\zeta)^k F\left(\tfrac{1}{2}, \tfrac{1}{2}, 1+k; \tfrac{1}{2}(1-\zeta)\right)\big],\end{aligned} \tag{4.12}$$

where

$$k = \tfrac{1}{2}(\sigma^2+1)^{1/2}, \qquad \zeta = \tanh(2\varphi).$$

4.2.4. General case. Since, for general values of M, the spectral problem is given by a differential equation with non-constant coefficients, its solution is not trivial. However, the reduction of the static problem to a well-known quantum mechanical problem suggests the use of a method that is commonly used in this field.

Observe that at $\varphi = \pm\infty$, (3.24) reduces to a differential equation with constant coefficients:

$$\left[\left(\alpha^2-\beta^2\right)\frac{d^2}{d\varphi^2} - 2\sigma\alpha\frac{d}{d\varphi} + \sigma^2 - \left(\alpha^2-\beta^2\right)\right]\gamma(\varphi) = 0. \tag{4.13}$$

This simplifies the problem considerably, since the behaviour of the solution at $\varphi = \pm\infty$ can be determined:

$$\begin{aligned}\gamma(\varphi) \sim C_1^{\pm}\exp&\left\{\left[\frac{\alpha\sigma}{\alpha^2-\beta^2} + \left(1+\frac{\beta^2\sigma^2}{(\alpha^2-\beta^2)^2}\right)^{1/2}\right]\varphi\right\} \\ &+C_2^{\pm}\exp\left\{\left[\frac{\alpha\sigma}{\alpha^2-\beta^2} - \left(1+\frac{\beta^2\sigma^2}{(\alpha^2-\beta^2)^2}\right)^{1/2}\right]\varphi\right\}.\end{aligned} \tag{4.14}$$

We can now apply 'scattering' techniques as in quantum mechanics, and 'connect' a solution at $\varphi = -\infty$ with a solution at $\varphi = +\infty$ (or vice versa). We start by choosing a value for σ for which a bounded solution at, say, $\varphi = -\infty$ exists. When at $\varphi = +\infty$ both types of solutions are bounded for this value of σ, it is always possible to 'connect' the solution at $\varphi = -\infty$ with a bounded solution at $\varphi = +\infty$. The corresponding value of σ therefore belongs to the (essential) spectrum. If, however, at $\varphi = +\infty$ only one type of solution is allowed for this value of σ then only in very special cases will it be possible to 'connect' the solutions. Therefore such a value of σ may be part of a 'discrete' spectrum. When no solutions are allowed, the value of σ is not part of the spectrum. The same argument can be applied starting at $\varphi = +\infty$. In this way we are able to calculate the essential

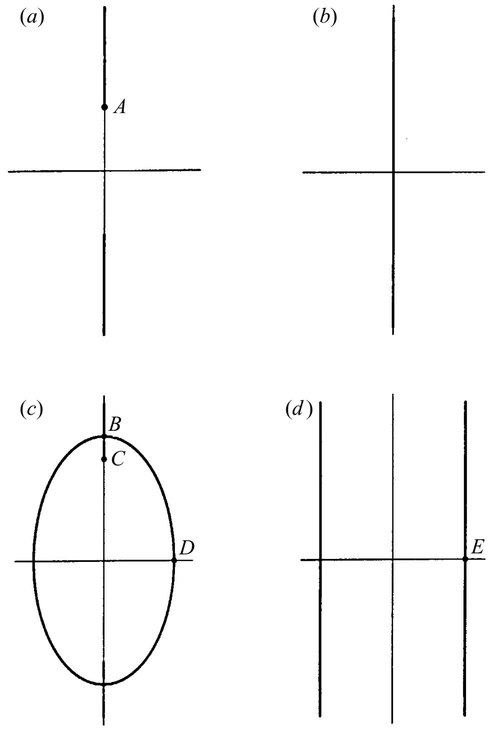

Figure 2. The spectrum for azimuthal modes: (a) $0 \leqslant |M| < 1$; (b) $|M| = 1$; (c) $1 < |M| < \infty$; (d) $|M| = \infty$. The indicated special points have the values $A = i(\beta^2 - \alpha^2)/|\beta|$, $B = i(\alpha^2 - \beta^2)^{1/2}|\alpha|/|\beta|$, $C = i(\alpha^2 - \beta^2)/|\beta|$, $D = (\alpha^2 - \beta^2)^{1/2}$ and $E = 1$.

spectrum, but we are only able to determine the region in which possible 'discrete' spectral values are located.

From the above argument it is found that the essential spectrum is given by

$$\sigma = \alpha n i \pm (\alpha^2 - \beta^2 - \beta^2 n^2)^{1/2}, \qquad n \in \mathbb{R}, \tag{4.15}$$

for which the eigenfunctions are purely oscillatory at $\varphi = \pm\infty$. When $|M| < 1$ ($\beta^2 > \alpha^2$), (4.15) describes part of the imaginary axis in the complex σ plane. For $|M| > 1$ ($\alpha^2 > \beta^2$), there is an additional ellipse (see Fig. 2). In the special cases $M = 0$, $M = 1$ and $M = \infty$ the spectrum as calculated previously is retrieved.

From the above argument, it is also found that for $|M| > 1$ ($\alpha^2 > \beta^2$), a 'discrete' spectrum may only exist in the area inside the ellipse. Furthermore, it can be shown that this 'discrete' spectrum cannot lie on the real and imaginary axes. Note that there need not be such a 'discrete' spectrum. For example, in the quantum mechanical problem of a particle moving in a potential well, there is both an essential and a discrete spectrum. However, in the similar problem of a particle moving around a potential barrier, there is only an essential spectrum.

Observe that spectrum (4.15), and, in fact, (4.13), does not depend on ρ. This means that the spectrum is infinitely degenerate. Since the spectrum does not depend on ρ, it must also correspond to the continuous spectrum of singular flux surface modes. This was also observed for the spectrum (4.2), and suggests that these modes are related to the well-known Alfvén and slow-magnetosonic continua. This will be explored further in Sec. 6.

Equation (3.24) can be transformed into an equation similar to (4.10). This enables us, for $\delta = 0$, to express its solutions in terms of hypergeometric functions. Introducing a new variable T, with

$$\gamma(\varphi) = \exp\left(\frac{\alpha\sigma}{\alpha^2 - \beta^2}\varphi\right) T(\varphi), \tag{4.16}$$

transforms (3.24) into

$$\frac{d^2T}{d\varphi^2} - \frac{1-\delta^2}{(c^2 + s^2 - \delta)^2} T = \left[\frac{\beta^2\sigma^2}{(\alpha^2 - \beta^2)^2} + 1\right] T. \tag{4.17}$$

Note that it is easily seen from this equation that the spectrum is symmetric with respect to the real and imaginary axes of the complex σ plane.

Equation (4.17) is the same kind of equation as we found for the static case, and so for $\delta = 0$ the solutions can be written down in terms of hypergeometric functions. This results in

$$\gamma(\varphi) = \exp\left(\frac{\alpha\sigma}{\alpha^2 - \beta^2}\varphi\right) (c^2 + s^2)^k \left\{C_1 F\left(\tfrac{1}{2} - k, \tfrac{1}{2} - k, 1 - k; \tfrac{1}{2}(1-\zeta)\right)\right.$$
$$\left. + C_2 (1-\zeta)^k F\left(\tfrac{1}{2}, \tfrac{1}{2}, 1 + k; \tfrac{1}{2}(1-\zeta)\right)\right\}, \tag{4.18}$$

where

$$k = \frac{1}{2}\left[1 + \frac{\beta^2\sigma^2}{(\alpha^2 - \beta^2)^2}\right]^{1/2}, \qquad \zeta = \tanh(2\varphi).$$

Since the behaviour of the hypergeometric functions at infinity is known, one can determine the spectrum from these solutions. The essential spectrum as described above is found and, furthermore, it can be shown that in this case no discrete spectrum exists.

In summary, in this section the essential spectrum for azimuthal modes has been determined. This spectrum is shown in Fig. 2. For $M = \infty$ and $M = 1$, the corresponding eigenfunctions were found and for $M = 0$ and for general M with $\delta = 0$ we were able to express the eigenfunctions in terms of hypergeometric functions. It was found from these expressions that a 'discrete' spectrum cannot exist when $\delta = 0$. Since the spectrum does not depend on ρ, it is infinitely degenerate and must coincide with the continuous spectrum of singular flux surface modes.

5. Spectrum of non-azimuthal modes

In this section the spectral problem for non-azimuthal modes ($k_z \neq 0$) is discussed. This problem is described by (3.28) and supplemented with the appropriate regularity conditions as described in Sec. 2. For non-azimuthal modes, the spectral problem is mathematically far more complex than the problem for azimuthal modes. Not only do we have two differential equations with non-constant coefficients, but these equations are also coupled, whereas for azimuthal modes we have two decoupled equations. Therefore, apart from two very special cases that are discussed in the Appendix, no explicit analytical solutions of (3.28) are found.

Nevertheless, we can determine the spectrum explicitly. Note that for $\varphi \to \infty$, the system (3.28) decouples into two ordinary differential equations with constant coefficients. Hence it is possible to find the solution of this system at infinity:

$$\begin{aligned}\begin{pmatrix}\xi(\varphi)\\ \tilde{\eta}(\varphi)\end{pmatrix} \sim{}& D_1^{\pm}\begin{pmatrix}0\\1\end{pmatrix}\exp\!\left(\frac{\sigma - iF_z}{\alpha-\beta}\varphi\right) + D_2^{\pm}\begin{pmatrix}0\\1\end{pmatrix}\exp\!\left(\frac{\sigma + iF_z}{\alpha+\beta}\varphi\right)\\ &+E_1^{\pm}\begin{pmatrix}1\\0\end{pmatrix}\exp\left\{\left[\frac{\alpha\sigma - i\beta F_z}{\alpha^2-\beta^2} + \left(1 + \frac{(\beta\sigma - i\alpha F_z)^2}{(\alpha^2-\beta^2)^2}\right)^{1/2}\right]\varphi\right\}\\ &+E_2^{\pm}\begin{pmatrix}1\\0\end{pmatrix}\exp\left\{\left[\frac{\alpha\sigma - i\beta F_z}{\alpha^2-\beta^2} - \left(1 + \frac{(\beta\sigma - i\alpha F_z)^2}{(\alpha^2-\beta^2)^2}\right)^{1/2}\right]\varphi\right\}.\end{aligned} \tag{5.1}$$

The solutions with $\tilde{\eta}$ polarization at infinity correspond to the parallel displacements in the case of azimuthal modes, whereas the solutions with ξ polarization correspond to the perpendicular displacements of that case. The introduction of a finite value for k_z couples the eigenmodes in the inner region, and changes the oscillatory behaviour whenever there is a longitudinal magnetic field component.

Since the behaviour of the solution at infinity is now known, we can again apply 'scattering' techniques to find (parts) of the spectrum. Thus we find an essential spectrum of the form

$$\sigma = \alpha n i \pm [\alpha^2 - \beta^2 - (\beta n - F_z)^2]^{1/2}, \qquad n \in \mathbb{R}. \tag{5.2}$$

This equation describes the same spectrum as (4.15), except that it is shifted along the imaginary axis by an amount $\alpha F_z/\beta$. In the absence of a poloidal magnetic field ($\beta = 0$, $\alpha = 1$), (5.2) reduces to

$$\sigma = in \pm (1 - F_z^2)^{1/2}, \tag{5.3}$$

so that the longitudinal magnetic field does not give rise to a shift of the spectrum but stabilizes these modes.

In order to investigate the spectrum further, the spectral problem is solved numerically for non-azimuthal modes in Fourier space. Note that (3.28) is a quadratic eigenvalue problem for the eigenvalue σ. To solve it numerically, the number of equations is doubled in order to obtain a linear eigenvalue problem. This is accomplished by introducing the following new variables:

$$\boldsymbol{\chi}_\perp \equiv \{\sigma - (\alpha+\beta)\,(\mathscr{D} + \mathscr{Q}) + iF_z\}\,\boldsymbol{\xi}_\perp, \tag{5.4}$$

where $\mathscr{Q}\boldsymbol{\xi}_\perp = \lambda\eta\mathbf{e}_\rho + \mu\eta\mathbf{e}_\varphi$. As a side-result, only first-order derivatives remain. However, when implementing the equations, we do not use the variables $\boldsymbol{\xi}_\perp$ and $\boldsymbol{\chi}_\perp$, but use ξ, $\tilde{\eta}$, χ_ξ and $(c^2+s^2)^{1/2}\chi_\eta$ in order to be consistent with the regularity conditions at infinity. In this way, we obtain a fourth-order, linear eigenvalue system for the spectral value σ.

When solving this new spectral problem numerically, the infinite interval $(-\infty, +\infty)$ is replaced by a finite interval $[-L, L]$, where L is a large number. The regularity conditions at infinity are replaced by the boundary conditions

$$\xi(-L) = \xi(L) = 0, \qquad \tilde{\eta}(-L) = \tilde{\eta}(L) = 0. \tag{5.5}$$

In this way, the original problem is approximated well, whenever L is large enough. Note that, owing to the finite domain, the dense set of spectral values belonging to the essential spectrum is discretized. This set of discrete values is then an approximation of the essential spectrum, and therefore we shall refer to the

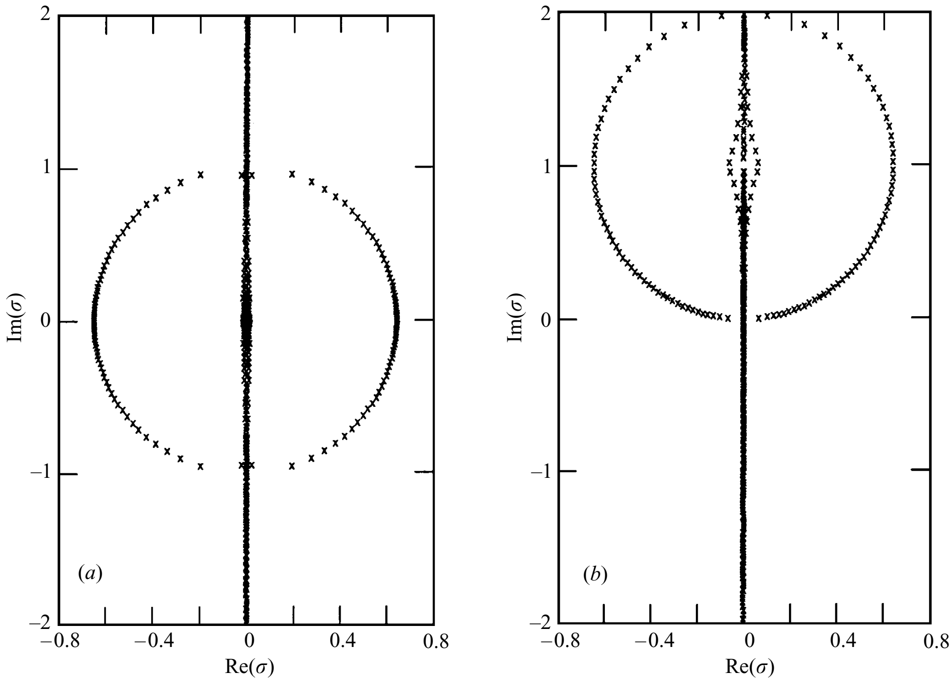

Figure 3. Spectrum of non-azimuthal modes for $F_z = 0$ (*a*) and $F_z = 0.64$ (*b*); $\alpha = 0.84$, $\beta = 0.54$, $\kappa = 1.00$, $\delta = 0$, $L = 300$ and 199 gridpoints are used.

spectral values of this set as belonging to the essential spectrum. Note that the existence of actual 'discrete' modes (on an unbounded domain) is not excluded a priori.

The problem on the finite interval $[-L, L]$ is discretized using finite elements in a way similar to that described in Nijboer *et al.* (1997 *b*). We use a mesh that is accumulated around the origin, allowing us to consider large values of L while keeping the number of gridpoints reasonably small. This gives rise to a large matrix eigenvalue problem which is solved using standerd techniques. A QR solver is used to calculate the complete spectrum and an Inverse Vector Iteration method to calculate single modes and the corresponding eigenfunctions. The merits of the two methods are described in Nijboer *et al.* (1997 *b*).

In Fig. 3 the numerically calculated spectrum for vanishing (*a*) and finite (*b*) values of $F_z = k_z B_z$ is shown. The spectrum agrees very well with the analytical results. Moreover, not only is the 'ellipse' of spectral values (5.2) found, but we also find that the complete imaginary axis is covered with spectral values just like for azimuthal modes. This means that the spectrum (4.2) is also retrieved for non-azimuthal modes.

Apart from the spectrum described above, no other spectral values were found, even when the parameters κ and δ were varied. In fact, when changing these parameters, the spectrum remained unchanged. Only a change of the value of F_z resulted in a shift of the complete spectrum (Fig. 3 *b*), which is in agreement with the analytical results. The off-axis modes, which are shown in the corresponding frame, are not converged. They tend towards the imaginary axis when L and/or the number of gridpoints are increased.

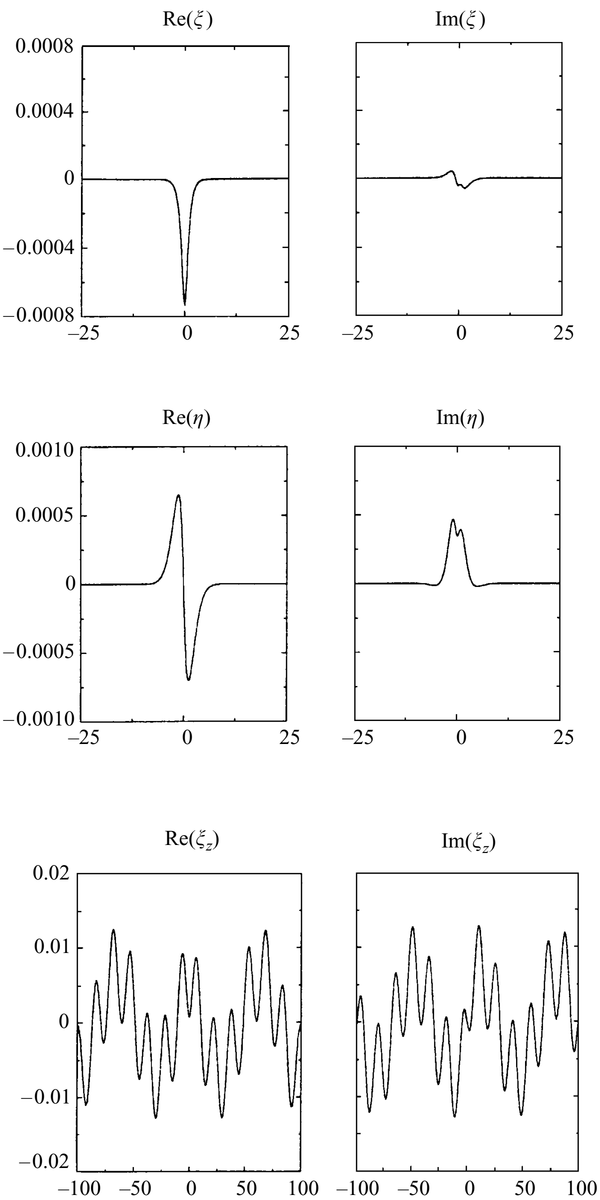

Figure 4. Eigenfunctions in Fourier space for the spectral value $\sigma = 0.12i$. The equilibrium parameters are the same as for Fig. 3(*a*), except that the number of gridpoints is 3200.

The fact that a variation of κ has no effect on the spectrum means that the spectrum is independent of ρ and hence infinitely degenerate. Therefore the continuous spectrum of singular flux surface modes also gives rise to these spectral values. Note, however, that, for non-azimuthal modes, the two continua are coupled.

An example of an eigenfunction in Fourier space is shown in Fig. 4. This is a mode that could not be computed analytically. Note that the components ξ and η fall off to zero for φ large enough. Only the component ξ_z does not fall off, but remains bounded. From this, we conclude two things. First, since ξ_z does not fall off, this mode belongs to the essential spectrum and, secondly, since it is the ξ_z component, this eigenfunction corresponds to that for parallel displacements of azimuthal modes. Similarly, we found that the eigenfunctions corresponding to modes on the 'ellipse' are in agreement with our analytical results and correspond to perpendicular displacements of azimuthal modes.

Finally, we mention that in the numerical study no modes were found for which all the variables fall off to zero at infinity. This means that all modes belong to the essential spectrum, and no discrete modes were found.

6. Continuous spectrum of singular flux surface modes

In this section the relation between the 'doubly' continuous spectrum as found in this paper and the 'classical' continuous spectrum is investigated. The 'classical' continuous spectrum is described by eigenfuncions for which the variations are mainly in the flux surfaces. Using this property, this classical spectrum can be described by ordinary differential equations, and hence is far easier to solve analytically than the rest of the spectrum. Here we investigate this part of the spectrum in physical space, and show how it is related to our results from Fourier space. For the calculation in physical space, we follow the same procedure as Hameiri and Hammer (1979).

The spectral problem in physical space is given by (3.8)–(3.11). Using the abbreviations

$$\tilde{c} \equiv \cosh\phi, \qquad \tilde{s} \equiv \sinh\phi, \tag{6.1}$$

the displacement vector can be projected on and perpendicular to the flux surfaces:

$$\xi_r = \frac{\tilde{c}\xi_x - \tilde{s}\xi_y}{(\tilde{c}^2 + \tilde{s}^2)^{1/2}}, \qquad \xi_\phi = \frac{\tilde{s}\xi_x + \tilde{c}\xi_y}{(\tilde{c}^2 + \tilde{s}^2)^{1/2}}. \tag{6.2}$$

In these variables, the spectral problem can be written as follows:

$$\mathsf{A}\begin{pmatrix} \partial_r\xi_r \\ \partial_r\pi \end{pmatrix} + \mathsf{B}\begin{pmatrix} \xi_r \\ \pi \end{pmatrix} + \mathsf{C}\begin{pmatrix} \xi_\phi \\ \xi_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{6.3}$$

$$\mathsf{D}\begin{pmatrix} \xi_r \\ \pi \end{pmatrix} + \mathsf{E}\begin{pmatrix} \xi_\phi \\ \xi_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.4}$$

The matrix operators $\mathsf{A}, \ldots, \mathsf{E}$ depend on the equilibrium, the spectral value σ and the derivatives in the flux surface. For the longitudinal dependence, Fourier modes are taken; hence the only differential operator in these matrices is ∂_ϕ.

Solving (6.4) for ξ_ϕ and ξ_z and substituting these solutions into (6.3) then gives a system for ξ_r and π. The 'classical' continuous spectrum is now found when the radial derivatives become large. This may happen whenever the matrix E is not invertible. E turns out to be diagonal, and hence the continuous spectrum is described by

$$E_{11}\xi_\phi = 0, \qquad E_{22}\xi_z = 0, \tag{6.5}$$

where E_{22} has constant coefficients and E_{11} has non-constant coefficients, which become constant at $\phi = \pm\infty$.

Demanding bounded solutions, we take an $e^{im\phi}$ dependence for ξ_z and the same dependence for ξ_ϕ at $\phi = \pm\infty$. This then yields the following spectral values:

$$\sigma = -i\alpha m \pm [\alpha^2 - \beta^2 - (\beta m + F_z)^2]^{1/2}, \tag{6.6}$$

which follows from E_{11}, and

$$\sigma = -i\alpha m \pm i(\beta m + F_z), \tag{6.7}$$

which follows from E_{22}. These expressions show that the 'classical' continuous spectrum for X-point flows becomes unstable for $M^2 > 1$. For closed flux-surface geometries, such unstable continua were also found by Hameiri and Hammer (1979) and Hellsten & Spies (1979).

The 'classical' continuous spectrum (6.6) and (6.7) is found as the special case $\rho \to \infty$ in the Fourier analysis of Sec. 5. However, the procedure of Section 5 was to obtain the 'doubly' continuous spectrum as the limit of the approximate spectrum for finite φ intervals. This approximate spectrum depends on ρ, but this dependence disappears in the limit when the φ interval becomes infinite. Therefore the 'doubly' continuous spectrum has to coincide with the 'classical' continuous spectrum. Also, note that although the spectrum does not depend on the parameter ρ, the eigenfunctions do depend on it.

The fact that the eigenfunctions ξ_ϕ and ξ_z become singular for the 'classical' continuous spectrum can also be shown with the Fourier method. In Fourier space the ρ dependence is parametrical, and therefore the eigenfunctions can be considered to have a delta-function dependence $\delta(\rho - \rho_0)$, where ρ_0 labels a specific flux surface. Using the Fourier transform (3.19), the eigenfunctions in physical space can be expressed in the eigenfunctions in Fourier space:

$$\xi_r = \frac{1}{2\pi(\cosh 2\phi)^{1/2}} \int \left\{ \frac{\cosh(\phi+\varphi)\,\xi - \sinh(\phi-\varphi)\,\eta}{(\cosh 2\varphi)^{1/2}} \right\} e^{ir\rho_0\cosh(\varphi+\phi)} \rho_0\, d\varphi, \tag{6.8}$$

$$\xi_\phi = \frac{1}{2\pi(\cosh 2\phi)^{1/2}} \int \left\{ \frac{\sinh(\phi-\varphi)\,\xi + \cosh(\phi+\varphi)\,\eta}{(\cosh 2\varphi)^{1/2}} \right\} e^{ir\rho_0\cosh(\varphi+\phi)} \rho_0\, d\varphi, \tag{6.9}$$

$$\xi_z = \frac{1}{2\pi} \int \hat{\xi}_z e^{ir\rho_0\cosh(\varphi+\phi)} \rho_0\, d\varphi, \tag{6.10}$$

where the z dependence has been ignored. The classical continuous spectrum is then retrieved in the limit $\rho_0 \to \infty$. Using the method of stationary phases (Copson 1965) then yields

$$\xi_r = \mathcal{O}(1), \tag{6.11}$$

$$\xi_\phi = \mathcal{O}(\rho_0^{1/2}), \tag{6.12}$$

$$\xi_z = \mathcal{O}(\rho_0^{1/2}). \tag{6.13}$$

Hence the eigenfunctions are polarized in the flux surfaces.

7. Conclusions

In this paper linear, incompressible plasma flows about a cylindrical X-point geometry have been considered. The equilibrium states of these flows have only one, translational, symmetry. Therefore they can be described by the modified Grad–Shafranov equation, which depends on profiles for the poloidal Alfvén Mach number and the Bernoulli function. Choosing a constant profile for the poloidal Alfvén

Mach number and a linear profile for the Bernoulli function gives a description of equilibria with elliptical flux surfaces (O points) or with hyperbolic flux surfaces (X points). Here the X-type equilibria are considered.

Perturbing the X-point equilibrium leads to a spectral problem in physical space, which is two-dimensional. The calculation of this spectrum in Fourier space greatly simplifies the spectral problem. It reduces the two-dimensional to a one-dimensional problem, leaving only derivatives along the field lines. In this way, the method stresses the dependence of the spectrum on the poloidal direction.

Since the flux surfaces of the X-point equilibria are open, the domain of the problem is infinite in essentially two directions. This allows for spectral sets that form points, curves or even areas in the complex plane. Here only curves were found, which are dense sets of eigenvalues. Hence local structures, like cluster spectra, are lost, while global structures, like the curves on which the spectral values lie, remain. The spectral values do not fill an area of the spectral plane, as in Lifschitz (1997 *b*), owing to the fact that the spectrum is infinitely degenerate, since it does not depend on ρ. This degeneracy also implies that the spectrum found here coincides with the classical continuous spectrum of singular flux surface modes.

For finite values of ρ, the spectrum is a dense set of spectral values. If the domain were to be bounded in physical or Fourier space, this would lead to a discrete spectrum. Note, however, that the discrete spectrum due to a bounded physical space is different from the discrete spectrum due to a bounded Fourier space. The continuous spectra that we find for the unbounded spaces, however, do coincide. The possibility of a discrete spectrum on an unbounded domain was not excluded a priori. This would have corresponded with modes for which all variables fall off to zero at infinity. However, no such modes were found.

For azimuthal modes, the essential spectrum was calculated and, for a number of special cases, also the corresponding eigenfunctions in Fourier space. The spectrum of parallel displacements corresponds to stable perturbations, whereas perturbations corresponding to the spectrum of perpendicular displacements become unstable when $M^2 > 1$, i.e. when the poloidal flow dominates over the poloidal magnetic field. In the limit of pure flow, the results are in agreement with Lifschitz (1997 *b*). It is also found that the essential spectrum depends neither on the geometry (expressed by the parameter δ) nor on the 'radial' Fourier mode ρ.

For non-azimuthal modes, the spectral problem is tackled numerically. The spectrum is basically the same as for azimuthal modes. All that changes is that the complete spectrum is shifted along the imaginary axis due to the longitudinal magnetic field. Hence, when $M^2 > 1$, the unstable modes are retrieved. These modes all belong to the essential spectrum, and do not depend on geometry (δ) or radial Fourier mode (ρ).

Since the spectrum for both azimuthal and non-azimuthal modes does not depend on the radial Fourier mode, it corresponds to the continuous spectrum of singular flux surface modes ($\rho \to \infty$). Hence the well-known Alfvén and slow-magnetosonic continua will correspond to these spectra. However, since the polarization of the modes is different from the one-dimensional case, one cannot label one kind of spectrum Alfvénic and the other slow magnetosonic. Since the equilibrium is two-dimensional, the Alfvén and slow magnetosonic continua will be coupled, and this coupling gives rise to the spectra found. Note that, for $M^2 > 1$, one of these spectra corresponds to unstable perturbations, and therefore to an unstable continuum.

Whenever the poloidal magnetic field is larger than the poloidal flow, the system is stable. This is in agreement with early results on the stabilty of static cusp geometries (Bernstein *et al.* 1958; Spalding 1971). On the other hand, whenever the poloidal flow dominates the poloidal magnetic field, the system becomes unstable, which agrees with the result of Lifschitz (1997*b*). Hence both known limits have been retrieved, and it has been shown that the transition from stable to unstable systems occurs at $M = 1$.

The results of this study should be of interest both for thermonuclear fusion research and for solar physics and astrophysics. X points arise in a tokamak owing, for example, to divertor action, or may be the result of magnetic reconnection due to tearing instability. The latter situation also applies for solar flares. Generally, the non-resistive behaviour of these X-point structures is considered to be stable. However, significant plasma flows do occur, and they alter the stability of the ideal plasma, as we have shown in this paper. Therefore, apart from the resistive instability that is usually considered, an ideal instability due to the plasma flow may arise and alter the plasma behaviour in these situations.

Acknowledgements

This work was performed as part of the research programme of the 'Stichting voor Fundamenteel Onderzoek der Materie' (FOM) with financial support from the 'Nederlandse Organisatie voor Wetenschappelijk Onderzoek' (NWO). It was supported by Grant DMS-9623033 of the National Science Foundation (NSF).

Appendix

As mentioned in Sec. 5, there are two special cases for which it is possible to solve the spectral problem (3.28) for non-azimuthal modes analytically: for a non-magnetized fluid ($\alpha = 1$, $\beta = 0$, $F_z = 0$) and for a static plasma with perturbations that do not vary with time ($\alpha = 0$, $\beta = 1$, $\sigma = 0$). For these cases, it is more convenient to use the variables ξ and η instead of ξ and $\tilde{\eta}$. Then, when $\alpha = 0$ or $\beta = 0$, (3.28) can be transformed into

$$\left(\frac{d^2}{d\varphi^2} + 2\mathscr{N}\frac{d}{d\varphi} + \mathscr{M} + d\right)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{A 1}$$

where

$$\mathscr{N} = \begin{pmatrix} 0 & \dfrac{1}{c^2+s^2} + \lambda \\ -\dfrac{1}{c^2+s^2} & \mu \end{pmatrix},$$

$$\mathscr{M} = \frac{4cs}{c^2+s^2}\begin{pmatrix} 0 & -\dfrac{1}{c^2+s^2} + \lambda \\ \dfrac{1}{c^2+s^2} & \mu \end{pmatrix} - \left(1 + \frac{1}{(c^2+s^2)^2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $\alpha = 0$ and $\beta = 1$, the transformation,

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = e^{iF_z\varphi}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \qquad d = -\sigma^2, \tag{A 2}$$

is applied, and for $\alpha = 1$ and $\beta = 0$,

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = e^{\sigma\varphi} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \qquad d = F_z^2. \tag{A 3}$$

After some tedious algebra, (A 1) transforms to

$$\left(\frac{d}{d\varphi} + \mathscr{U}\right)\left(\frac{d}{d\varphi} + \mathscr{V}\right)\mathbf{u} + d\mathbf{u} = 0, \tag{A 4}$$

where $\mathbf{u} \equiv (u_1, u_2)^{\mathrm{T}} = \mathscr{W}(w_1, w_2)^{\mathrm{T}}$ and

$$\mathscr{U} = \begin{pmatrix} \dfrac{2cs\left[c^2+s^2-\delta-(1-\delta^2)\kappa^2\right]}{(c^2+s^2-\delta)\left[c^2+s^2-\delta+(1-\delta^2)\kappa^2\right]} & -\dfrac{2(1-\delta^2)(1-\delta^2)^{1/2}\kappa^2}{(c^2+s^2-\delta)\left[c^2+s^2-\delta+(1-\delta^2)\kappa^2\right]} \\ \dfrac{2\delta}{(1-\delta^2)^{1/2}} & \dfrac{2cs}{c^2+s^2-\delta} \end{pmatrix},$$

$$\mathscr{V} = \begin{pmatrix} \dfrac{2cs}{c^2+s^2-\delta} & 0 \\ \dfrac{2-2\delta(c^2+s^2)}{(1-\delta^2)^{1/2}(c^2+s^2-\delta)} & -\dfrac{2cs}{c^2+s^2-\delta} \end{pmatrix},$$

$$\mathscr{W} = \begin{pmatrix} \dfrac{(1-\delta^2)^{1/2}}{(c^2+s^2)^{1/2}(c^2+s^2-\delta)^{1/2}} & \dfrac{2cs(1-\delta^2)^{1/2}}{(c^2+s^2)^{1/2}(c^2+s^2-\delta)^{1/2}} \\ \dfrac{-2cs}{(c^2+s^2)^{1/2}(c^2+s^2-\delta)^{1/2}} & \dfrac{1-\delta(c^2+s^2)}{(c^2+s^2)^{1/2}(c^2+s^2-\delta)^{1/2}} \end{pmatrix}.$$

When $\delta = 0$, the matrix $\mathscr{U}$ is upper-triangular, and hence, when $d = 0$, (A 4) can be integrated. This yields the following solutions:

$$(u_1, u_2)_1 = \left(\frac{1}{(c^2+s^2)^{1/2}}, -2(c^2+s^2)^{1/2}\int_{\psi_3}^{\varphi}\frac{1}{(c^2+s^2)^2}\,d\varphi^*\right), \tag{A 5}$$

$$(u_1, u_2)_2 = \left(0, (c^2+s^2)^{1/2}\right), \tag{A 6}$$

$$\begin{aligned}(u_1, u_2)_3 = \Bigg(&\frac{1}{(c^2+s^2)^{1/2}}\int_{\psi_2}^{\varphi}\frac{c^2+s^2}{c^2+s^2+\kappa^2}\,d\varphi^*, \\ &-2(c^2+s^2)^{1/2}\int_{\psi_3}^{\varphi}\frac{1}{(c^2+s^2)^2}\int_{\psi_2}^{\varphi^*}\frac{c^2+s^2}{c^2+s^2+\kappa^2}\,d\varphi^{**}\,d\varphi^*\Bigg),\end{aligned} \tag{A 7}$$

$$(u_1, u_2)_4 = \left(\frac{2\kappa^2}{(c^2+s^2)^{1/2}} \int_{\psi_2}^{\varphi} \frac{c^2+s^2}{c^2+s^2+\kappa^2} \int_{\psi_1}^{\varphi^*} \frac{1}{(c^2+s^2)^2}\, d\varphi^{**}\, d\varphi^*, \right.$$

$$(c^2+s^2)^{1/2} \int_{\psi_3}^{\varphi} \left[\frac{1}{c^2+s^2} - \frac{4\kappa^2}{(c^2+s^2)^2} \int_{\psi_2}^{\varphi^*} \frac{c^2+s^2}{c^2+s^2+\kappa^2} \right.$$

$$\left. \left. \times \int_{\psi_1}^{\varphi^{**}} \frac{1}{(c^2+s^2)^2}\, d\varphi^{***}\, d\varphi^{**} \right] d\varphi^* \right), \quad \text{(A 8)}$$

where $\psi_i = \pm\infty$, $i = 1, 2, 3$.

In the case of a static plasma with time-independent perturbations ($\alpha = 0$, $\beta = 1$, $\sigma = 0$), a linear combination of the solutions (A 5) and (A 7) gives rise to an allowed solution. Hence, in this case, $\sigma = 0$ is an eigenvalue. This spectral value corresponds to the spectrum (4.2) for non-azimuthal modes.

In the case of a non-magnetized fluid ($\alpha = 1$, $\beta = 0$, $F_z = 0$) two types of solutions are allowed. Again, the same combination of solutions (A 5) and (A 7) now yields that all pure imaginary values of σ belong to the spectrum. Again, these modes correspond to the spectrum (4.2) for non-azimuthal modes. However, apart from these solutions, the solution (A 8) is also allowed when we choose $\psi_i = +\infty$ for $\mathrm{Re}(\sigma) > 0$ and $\psi_i = -\infty$ for $\mathrm{Re}(\sigma) < 0$, $i = 1, 2, 3$. This yields the spectral values

$$\mathrm{Re}(\sigma) = \pm 1, \quad \text{(A 9)}$$

which is the spectrum (5.2). In this case, the eigenfunctions are in agreement with Lifschitz (1997 *b*). Note that the spectrum is essential, and no discrete spectrum exists.

References

Agim, Y. Z. and J. A. Tataronis 1985 General two-dimensional magnetohydrodynamic equilibria with mass flow. *J. Plasma Phys.* **34**, 337–360.

Bayly, B. J. 1986 Three-dimensional instability of elliptical flow. *Phys. Rev. Lett.* **57**, 2160–2163.

Berge, G. 1997 A Lyapunov functional for stability of ideal magnetohydrodynamic systems in arbitrary motion. *Phys. Plasmas* **4**, 1201–1212.

Bernstein, I. B., Frieman, E. A., Kruskal, M. D. and Kulsrud R. M. 1958 An energy principle for hydromagnetic stability problems. *Proc. R. Soc. Lond.* **A 224**, 17–40.

Copson, E. T. 1965 *Asymptotic Expansions.* Cambridge University Press.

Dewar, R. L., Grimm, R. C., Johnson, J. L., Frieman, E. A., Greene, J. M. and Rutherford, P. H. 1974 Long-wavelength kink instabilities in low-pressure, uniform axial current, cylindrical plasmas with elliptic cross sections. *Phys. Fluids* **17**, 930–938.

Frieman, E. and Rotenberg, M. 1960 On the hydromagnetic stability of stationary equilibria. *Rev. Mod. Phys.* **32**, 898–902.

Gajewski, R. 1972 Magnetohydrodynamic equilibrium of an elliptical plasma cylinder. *Phys. Fluids* **15**, 70–74.

Goedbloed, J. P. 1975 Spectrum of ideal magnetohydrodynamics of axisymmetric toroidal systems. *Phys. Fluids* **18**, 1258–1268.

Goedbloed, J. P. and Lifschitz, A. E. 1997 Stationary symmetric magnetohydrodynamic flows. *Phys. Plasmas* **4**, 3544–3564.

Hameiri, E. and Hammer, J. H. 1979 Unstable continuous spectrum in magnetohydrodynamics. *Phys. Fluids* **22**, 1700–1706.

Hellsten, T. A. K. and Spies, G. O. 1979 Continuous magnetohydrodynamic spectrum in axially symmetric rotating plasmas. *Phys. Fluids* **22**, 743–746.

Lagnado, R. R., Phan-Thien, N. and Leal, L. G. 1984 The stability of two-dimensional flows. *Phys. Fluids* **27**, 1094–1101.

Landau, L. D. and Lifschitz, E. M. 1977 *Quantum Mechanics (Non-relativistic Theory)*. Pergamon Press, Oxford.

Lifschitz, A. 1989 *Magnetohydrodynamics and Spectral Theory*. Kluwer, Dordrecht.

Lifschitz, A. 1995 Exact description of the spectrum of elliptical vortices in hydrodynamics and magnetohydrodynamics. *Phys. Fluids* **A7**, 1626–1636.

Lifschitz, A. 1997 *a* A nonlinear spectral problem with periodic coefficients occurring in magnetohydrodynamic stability theory. *Proceedings of the 1995 International Workshop on Operator Theory and Applications* (ed. R. Mennicken and C. Tretter), pp. 97–117 Birkhäuser, Boston.

Lifschitz, A. 1997 *b* On the spectrum of hyperbolic flows. *Phys. Fluids* **9**, 2864–2871.

Lifschitz, A. and Goedbloed, J. P. 1997 Transonic magnetohydrodynamic flows. *J. Plasma Phys.* **58**, 61–100.

Nijboer, R. J., Lifschitz, A. E. and Goedbloed, J. P. 1997 *a* Spectrum and stability of a rigidly rotating compressible plasma. *J. Plasma Phys.* **58**, 101–121.

Nijboer, R. J., van der Holst, B., Poedts, S. and Goedbloed, J. P. 1997 *b* Calculating magnetohydrodynamic flow spectra. *Comp. Phys. Commun.* **106**, 39–52.

Sahli, A., Cambon, C. and Speziale, C. G. 1997 Linear stability analysis of plane quadratic flows in a rotating frame with applications to modelling. *Phys. Fluids* **9**, 2300–2309.

Spalding, I. 1971 Cusp containment. *Advances in Plasma Physics*, Vol. 4 (ed. A. Simon and W. B. Thompson). Interscience Publishers, New York.

Spies, G. O. 1978 Magnetohydrodynamic spectrum of instabilities due to plasma rotation. *Phys. Fluids* **21**, 580–587.